Central Limit Theorem

Random variables $x_1(\zeta), x_2(\zeta), \ldots, x_M(\zeta)$

- (a) are mutually independent and
- (b) have the same distribution, and
- (c) the mean and variance of each random variable exist and are finite

Then, the distribution of the normalized sum

$$y_M(\zeta) = \frac{\sum_{k=1}^{M} x_k(\zeta) - \mu_{y_M}}{\sigma_{y_M}}$$

approaches that of a normal random variable with zero mean and unit standard deviation as $M \to \infty$. 
The sample space $S$, the probabilities $Pr\{\zeta_k\}$, and the sequences $x(n, \zeta_k), -\infty < n < \infty$, constitute a discrete-time stochastic process or random sequence.
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The set of all possible sequences $\{x(n, \zeta)\}$ is called an ensemble, and each individual sequence $x(n, \zeta_k)$, corresponding to a specific value of $\zeta = \zeta_k$, is called a realization or a sample sequence of the ensemble.
Random Process
Random Process

- $x(n, \zeta)$ is a random variable if $n$ is fixed and $\zeta$ is a variable.
- $x(n, \zeta)$ is a sample sequence if $\zeta$ is fixed and $n$ is a variable.
- $x(n, \zeta)$ is a number if both $n$ and $\zeta$ are fixed.
- $x(n, \zeta)$ is a stochastic process if both $n$ and $\zeta$ are variables.
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1. The frequency of occurrence of various signal amplitudes, described by the probability distribution of samples. (histogram)

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3. The existence of “cycles” or quasi-periodic patterns, obtained from the signal power spectrum.

4. Indications of variability in the mean, variance, probability density, or spectral content.
The $k$th-order cdf:

$$F_x(x_1, \ldots, x_k; n_1, \ldots, n_k) = Pr\{x(n_1) \leq x_1, \ldots, x(n_k) \leq x_k\}$$

or the $k$th-order pdf:

$$f_x(x_1, \ldots, x_k; n_1, \ldots, n_k) = \frac{\partial^2 F_x(x_1, \ldots, x_k; n_1, \ldots, n_k)}{\partial x_{R1} \partial x_{I1} \cdots \partial x_{Rk} \partial x_{Ik}}$$

needs to be known for every value of $k \geq 1$ and for all instances $n_1, n_2, \ldots, n_k$. 
The second-order statistic of $x(n)$ at time $n$ is specified by its mean value $\mu_x(n)$ and its variance $\sigma_x^2(n)$, defined by

- $\mu_x(n) = E\{x(n)\} = E\{x_R(n) + jx_I(n)\}$
- $\sigma_x^2(n) = E\{|x(n) - \mu_x(n)|^2\} = E\{|x(n)|^2\} - |\mu_x(n)|^2$

respectively.

In general, both $\mu_x(n)$ and $\sigma_x(n)$ are deterministic sequences.
Autocorrelation and Autocovariance

The second-order statistics of $x(n)$ at two different times $n_1$ and $n_2$ are given by the two-dimensional autocorrelation (or autocovariance) sequences.

- $r_{xx}(n_1, n_2) = E\{x(n_1)x^*(n_2)\}$
- $\gamma_{xx}(n_1, n_2) = r_{xx}(n_1, n_2) - \mu_x(n_1)\mu_x^*(n_2)$
Cross-correlation and Cross-covariance

- \( r_{xy}(n_1, n_2) = E\{x(n_1)y^*(n_2)\} \)
- \( \gamma_{xy}(n_1, n_2) = r_{xy}(n_1, n_2) - \mu_x(n_1)\mu_y^*(n_2) \)
A sequence of independent random variables. If all random variables have the same pdf $f(x)$ for all $n_k$, then $x(n)$ is called an IID (independent and identically distributed) random sequence.

An uncorrelated process

$$\gamma_x(n_1, n_2) = 0 \quad \text{if} \quad n_1 \neq n_2$$
An orthogonal process

\[ r_x(n_1, n_2) = 0 \text{ if } n_1 \neq n_2 \]

A wise-sense cyclostationary process

\[ \mu_x(n) = \mu_x(n + N), \forall n \]

\[ r_x(n_1, n_2) = r_x(n_1 + N, n_2 + N), \forall n_1, n_2 \]
Stationarity

A stochastic process \( x(n) \) is called stationary of order \( N \) if

\[
f_x(x_1, \ldots, x_N; n_1, \ldots, n_N) = f_x(x_1, \ldots, x_N; n_1 + k, \ldots, n_N + k)
\]

for any value of \( k \).

If \( x(n) \) is stationary for all orders \( N = 1, 2, \ldots \), it is said to be strict-sense stationary (SSS).

- An IID sequence is SSS.
- If a stochastic process \( x(n) \) is stationary up to order 2, it is said to be wide-sense stationary (WSS).
A random signal $x(n)$ is called wide-sense stationary (WSS) if

1. Its mean is a constant independent of $n$,
2. Its variance is also a constant independent of $n$,
3. Its autocorrelation depends only on the distance $l = n_1 - n_2$, called lag.
A random signal $x(n)$ is called wide-sense stationary (WSS) if

1. $E\{x(n)\} = \mu_x$
2. $Var[x(n)] = \sigma_x^2$
3. $r_x(n_1, n_2) = r_x(n_1 - n_2) = r_x(l) = E\{x(n + l)x^*(n)\} = E\{x(n)x^*(n - l)\}$
Can we infer the statistical characteristics of the process from a single realization?

Ergodicity implies that all the statistical information can be obtained from any single representative member of the ensemble.
Time average:

\[
\langle (\cdot) \rangle = \lim_{N \to \infty} \frac{1}{2N + 1} \sum_{n=-N}^{N} (\cdot)
\]

- Mean value = \( \langle x(n) \rangle \)
- Mean square = \( \langle |x(n)|^2 \rangle \)
- Variance = \( \langle |x(n) - \langle x(n) \rangle|^2 \rangle \)
- Autocorrelation = \( \langle x(n)x^*(n-l) \rangle \)
- Autocovariance = \( \langle [x(n) - \langle x(n) \rangle][x(n-l) - \langle x(n) \rangle]^* \rangle \)
- Cross-correlation = \( \langle x(n)y^*(n-l) \rangle \)
- Cross-covariance = \( \langle [x(n) - \langle x(n) \rangle][y(n-l) - \langle y(n) \rangle]^* \rangle \)
A random signal $x(n)$ is called ergodic if its ensemble averages equal appropriate time averages.

If the process is stationary and ergodic, then all statistical information can be derived from only one typical realization of the process.
Ergodic Random Processes

A random process $x(n)$ is ergodic in the mean if

$$\langle x(n) \rangle = E\{x(n)\}$$

A random process $x(n)$ is ergodic in correlation if

$$\langle x(n)x^*(n - l) \rangle = E\{x(n)x^*(n - l)\}$$

- That is, $\langle x(n) \rangle$ is a constant and $\langle x(n)x^*(n - l) \rangle$ is a function of $l$. If $x(n)$ is ergodic in both mean and variance, it is WSS. Only stationary signals can be ergodic.
- On the other hand, WSS does not imply ergodicity of any kind.
• Power spectral density

• White noise
  A random sequence \( w(n) \) is called a (second-order) white noise process with mean \( \mu_w \) and variance \( \sigma_w^2 \), if and only if

\[
E\{w(n)\} = \mu_w
\]

and

\[
r_w(l) = E\{w(n)w^*(n - l)\} = \sigma_w^2 \delta(l)
\]

\[
R_w(e^{j\omega}) = \sigma_w^2
\]
An $M$-dimensional real-valued random vector

$$\mathbf{x}(\zeta) = [x_1(\zeta), x_2(\zeta), \ldots, x_M(\zeta)]^T$$

is completely characterized by its joint cdf:

$$F_{\mathbf{x}}(x_1, \ldots, x_M) = \Pr\{x_1(\zeta) \leq x_1, \ldots, x_M(\zeta) \leq x_M\} = \Pr\{\mathbf{x}(\zeta) \leq \mathbf{x}\}$$

or by its joint pdf:

$$f_{\mathbf{x}}(\mathbf{x}) = \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_M} F_{\mathbf{x}}(\mathbf{x})$$
An $M$-dimensional complex-valued random vector
\[ x(\zeta) = x_R(\zeta) + jx_I(\zeta) \]
is completely characterized by its joint cdf:
\[ F_x(x) = Pr\{x(\zeta) \leq x\} = Pr\{x_R(\zeta) \leq x_R, x_I(\zeta) \leq x_I\} \]
or by its joint pdf:
\[ f_x(x) = \frac{\partial}{\partial x_{R1}} \frac{\partial}{\partial x_{I1}} \cdots \frac{\partial}{\partial x_{RM}} \frac{\partial}{\partial x_{IM}} F_x(x) \]
Mean vector: $\mu_x = E\{x\} = [\mu_1, \mu_2, \ldots, \mu_M]^T$
Statistical Description

- Mean vector: $\mu_x = E\{x\} = [\mu_1, \mu_2, \ldots, \mu_M]^T$
- Auto-correlation matrix:

$$R_x = E\{xx^H\} = \begin{bmatrix} r_{11} & \cdots & r_{1M} \\ \vdots & \ddots & \vdots \\ r_{M1} & \cdots & r_{MM} \end{bmatrix}$$

where $r_{ij} = E\{x_i x_j^*\} = r_{ji}^*$. 
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$$\Gamma_x = E\{(x - \mu_x)(x - \mu_x)^H\}$$
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\Gamma_x = R_x - \mu_x\mu_x^H
\]
Cross-correlation matrix:

\[ R_{xy} = E\{x y^H\} = \begin{bmatrix}
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    \vdots & \ddots & \vdots \\
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\[ \Gamma_{xy} = R_{xy} - \mu_x \mu_y^H \]
If two random vectors $\mathbf{x}$ and $\mathbf{y}$ are uncorrelated

$$\Gamma_{xy} = 0 \Rightarrow R_{xy} = \mu_x \mu_y^H$$
- If two random vectors $x$ and $y$ are uncorrelated
  \[ \Gamma_{xy} = 0 \Rightarrow R_{xy} = \mu_x \mu_y^H \]

- If two random vectors $x$ and $y$ are orthogonal
  \[ R_{xy} = 0 \]
The correlation matrix of a random vector \( x \) is conjugate symmetric or Hermitian

\[
R_x = R_x^H
\]

The correlation matrix of a random vector \( x \) is nonnegative definite

\[
w^H R_x w \geq 0
\]

The eigenvalues \( \{ \lambda_i \}_{i=1}^M \) of correlation matrix \( R_x \) are real and nonnegative.

If \( R_x \) is positive definite, then \( \lambda_i > 0 \) for all \( 1 \leq i \leq M \).

If the eigenvalues \( \{ \lambda_i \}_{i=1}^M \) are distinct, then the corresponding eigenvectors are orthogonal to one another, that is,

\[
\lambda_i \neq \lambda_j \Rightarrow q_i^H q_j = 0, \quad \text{for } i \neq j
\]
The correlation matrix of a random vector $\mathbf{x}$ is conjugate symmetric or Hermitian

$$R_\mathbf{x} = R_\mathbf{x}^H$$

The correlation matrix of a random vector $\mathbf{x}$ is nonnegative definite

$$\mathbf{w}^H R_\mathbf{x} \mathbf{w} \geq 0$$
General Correlation Matrices

- The correlation matrix of a random vector $\mathbf{x}$ is conjugate symmetric or Hermitian
  \[ \mathbf{R}_x = \mathbf{R}_x^H \]

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  \[ \lambda_i \neq \lambda_j \Rightarrow \mathbf{q}_i^H \mathbf{q}_j = 0, \text{ for } i \neq j \]
Spectral Decomposition

Let \( \{q_i\}_{i=1}^M \) be an orthonormal set of eigenvectors corresponding to the distinct eigenvalues \( \{\lambda_i\}_{i=1}^M \) of an \( M \times M \) correlation matrix \( R \).

Then \( R \) can be diagonalized as

\[
\Lambda = Q^H R Q
\]

where the orthonormal matrix \( Q = [q_1 \cdots q_M] \) is known as an eigen-matrix and \( \Lambda \) is an \( M \times M \) diagonal eigenvalue matrix, that is,

\[
\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_M).
\]

The trace of \( R \) is the summation of all eigenvalues

\[
\text{tr}(R) = \sum_{i=1}^M \lambda_i
\]
A stochastic process can be represented as a random vector

\[ x(n) = [x(n), x(n-1), \cdots, x(n-M+1)]^T \]
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\[ x(n) = [x(n), x(n - 1), \cdots, x(n - M + 1)]^T \]
Mean: \( \mu_x(n) = [\mu_x(n), \mu_x(n - 1), \cdots, \mu_x(n - M + 1)]^T \)
A stochastic process can be represented as a random vector
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Mean: \[ \mu_x(n) = [\mu_x(n), \mu_x(n-1), \cdots, \mu_x(n-M+1)]^T \]

Correlation:
\[ R_x(n) = \begin{bmatrix}
    r_x(n, n) & \cdots & r_x(n, n-M+1) \\
    \vdots & \ddots & \vdots \\
    r_x(n-M+1, n) & \cdots & r_x(n-M+1, n-M+1)
\end{bmatrix} \]
Correlation Matrices from Random Processes

- Correlation matrices of stationary processes $R_x$

$$R_x(n) = \begin{bmatrix}
  r_x(0) & r_x(1) & \cdots & r_x(M - 1) \\
  r_x^*(1) & r_x(0) & \cdots & r_x(M - 2) \\
  \vdots & \vdots & \ddots & \vdots \\
  r_x^*(M - 1) & r_x^*(M - 2) & \cdots & r_x(0)
\end{bmatrix}$$

- $R_x$ is Hermitian and Toeplitz.
Whitening and Innovations Representation

- **Whitening** – To represent a random vector (or sequence) $x$ with a linearly equivalent vector (or sequence) consisting of uncorrelated components $w$.

If $x$ is a correlated random vector and if $A$ is a nonsingular matrix, then the linear transformation $w = Ax$ results in a random vector $w$ that contains the same "information" as $x$, and hence random vectors $x$ and $w$ are said to be linearly equivalent. Furthermore, if $w$ has uncorrelated components and $A$ is lower-triangular, then each component $w_i$ of $w$ can be thought of as adding "new" information (or innovation) to $w$.
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- If $\mathbf{x}$ is a correlated random vector and if $\mathbf{A}$ is a nonsingular matrix, then the linear transformation $\mathbf{w} = \mathbf{A}\mathbf{x}$ results in a random vector $\mathbf{w}$ that contains the same “information” as $\mathbf{x}$, and hence random vectors $\mathbf{x}$ and $\mathbf{w}$ are said to be linearly equivalent.
Whitening and Innovations Representation

- Whitening – To represent a random vector (or sequence) \( x \) with a linearly equivalent vector (or sequence) consisting of uncorrelated components \( w \).

- If \( x \) is a correlated random vector and if \( A \) is a nonsingular matrix, then the linear transformation \( w = Ax \) results in a random vector \( w \) that contains the same “information” as \( x \), and hence random vectors \( x \) and \( w \) are said to be linearly equivalent.

- Furthermore, if \( w \) has uncorrelated components and \( A \) is lower-triangular, then each component \( w_i \) of \( w \) can be thought of as adding “new” information (or innovation) to \( w \).
Let $Q_x$ be the eigenmatrix of $x$. Then $Q_x^H$ is the linear transformation matrix $A$.

The variances of random variables $w_i$, $i = 1, \ldots, M$, are equal to the eigenvalues of $x$.

Since the transformation matrix $A = Q_x^H$ is orthonormal, the transformation is called an orthonormal transformation.
Orthogonal Transformation
Isotropic Transformation

\[ y = \Lambda_x^{-1/2} w \quad \text{and} \quad R_y = I \]