ELC 5396: Digital Communications

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1. FIR Modeling of Doubly Spread Channels

2. Diversity Techniques

3. Space Diversity-on-Receive Systems
FIR Modeling of Doubly Spread Channels

- Impulse response of baseband channel \( \tilde{h}(\tau; t) \) and its transfer function \( \tilde{H}(f; t) \)
FIR Modeling of Doubly Spread Channels

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- $y(t) = \int_{-\infty}^{\infty} h(\tau; t)x(t-\tau)d\tau$
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- Impulse response of baseband channel $\tilde{h}(\tau; t)$ and its transfer function $\tilde{H}(f; t)$

- $y(t) = \int_{-\infty}^{\infty} h(\tau; t)x(t - \tau)d\tau$

- $x(t - \tau) = \sum_{n=-\infty}^{\infty} x(t - nT_s)\text{sinc}(\frac{\tau}{T_s} - n)$
FIR Modeling of Doubly Spread Channels

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$$y(t) = \sum_{n=-\infty}^{\infty} x(t - nT_s) \left[ \int_{-\infty}^{\infty} h(\tau; t) \text{sinc}\left(\frac{\tau}{T_s} - n\right) d\tau \right]$$

$$= \sum_{n=-\infty}^{\infty} x(t - nT_s) c_n(t)$$
FIR Modeling of Doubly Spread Channels

- FIR model of a time-varying channel.
- For time-varying Rayleigh fading channels, $c_n(t)$ is zero-mean complex Gaussian process.
- For WSS channels, $\mathbb{E}[|c_n(t)|^2] \approx T_s^2 p(n\tau)$, where $p(n\tau)$ is a discrete version of the power-delay profile.
The multipath fading phenomenon as an inherent characteristic of a wireless channel.

**Diversity:**
If several replicas of the information-bearing signal can be transmitted simultaneously over independently fading channels, then there is a good likelihood that at least one of the received signals will not be severely degraded by channel fading.
1. Frequency diversity – Transmission of same signal at different frequencies (frequency separation should be larger than the coherence bandwidth of the channel).
Diversity Techniques

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2. Time diversity – Transmission of same signal sequence at different times (time separation should be larger than the coherence time of the channel). Time diversity may be likened to the use of a repetition code for error-control coding.
Diversity Techniques

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3. Space diversity – Several receiving antennas spaced sufficiently far apart (spatial separation should be sufficiently large to reduce correlation between diversity branches, e.g., $> 10\lambda$).
Polarization diversity – Only two diversity branches are available. Not widely used.
Diversity Techniques

1. **Polarization diversity** – Only two diversity branches are available. Not widely used.

2. **Multipath diversity** – Signal replicas received at different delays (RAKE receiver in CDMA) or via different angles of arrival (directional antennas at the receiver). Equalization in a TDM/TDMA system provides similar performance as multipath diversity.
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   - Signal replicas received at different delays (RAKE receiver in CDMA)
   - Signal replicas received via different angles of arrival (directional antennas at the receiver)
   - Equalization in a TDM/TDMA system provides similar performance as multipath diversity.
Receive diversity – The use of a single transmit antenna and multiple receive antennas.
Space Diversity


2. **Transmit diversity** – The use of multiple transmit antennas and a single receive antenna.
Space Diversity


2. **Transmit diversity** – The use of multiple transmit antennas and a single receive antenna.

3. **Diversity on both transmit and receive** – The use of multiple antennas at both the transmitter and receiver.
Selection Combing

![Diagram of selection combing](image-url)
Space Diversity-on-Receive Systems

Maximal-Ratio Combing and Equal-Gain Combing

\[ \begin{align*}
1 & \xrightarrow{\tilde{x}_1(t)} \text{Receiver 1} \\
2 & \xrightarrow{\tilde{x}_2(t)} \text{Receiver 2} \\
& \vdots \\
N_r & \xrightarrow{\tilde{x}_{N_r}(t)} \text{Receiver } N_r \\
\end{align*} \]

\[ \xrightarrow{\text{Linear combiner}} \text{Output} \]
BER vs. SNR in a flat fading channel

In a flat fading channel (or narrowband system), the CIR (channel impulse response) reduces to a single impulse scaled by a time-varying complex coefficient. The received (equivalent lowpass) signal is of the form

\[ r(t) = a(t)e^{j\phi(t)}s(t) + n(t) \]

We assume that the phase changes “slowly” and can be perfectly tracked

\[ => \text{important for coherent detection} \]
BER vs. SNR (cont.)

We assume:

- the time-variant complex channel coefficient changes slowly (=> constant during a symbol interval)
- the channel coefficient magnitude (= attenuation factor) $a$ is a Rayleigh distributed random variable
- coherent detection of a binary PSK signal (assuming ideal phase synchronization)

Let us define instantaneous SNR and average SNR:

$$
\gamma = a^2 \frac{E_b}{N_0} \quad \gamma_0 = E\left\{a^2\right\} \cdot \frac{E_b}{N_0}
$$
BER vs. SNR (cont.)

Since

\[ p(a) = \frac{2a}{E\{a^2\}} e^{-a^2/E\{a^2\}} \quad a \geq 0, \]

using

\[ p(\gamma) = \frac{p(a)}{|d\gamma/da|} \]

we get

\[ p(\gamma) = \frac{1}{\gamma_0} e^{-\gamma/\gamma_0} \quad \gamma \geq 0. \]
The average bit error probability is

\[ P_e = \int_0^\infty P_e(\gamma) p(\gamma) d\gamma \]

where the bit error probability for a certain value of \( \alpha \) is

\[ P_e(\gamma) = Q\left(\sqrt{2\frac{\alpha^2 E_b}{N_0}}\right) = Q\left(\sqrt{2\gamma}\right). \]

We thus get

\[ P_e = \int_0^\infty Q\left(\sqrt{2\gamma}\right) \frac{1}{\gamma_0} e^{-\gamma/\gamma_0} d\gamma = \frac{1}{2} \left( 1 - \sqrt{\frac{\gamma_0}{1+\gamma_0}} \right). \]
Approximation for large values of average SNR is obtained in the following way. First, we write

\[ P_e = \frac{1}{2} \left( 1 - \sqrt{\frac{\gamma_0}{1+\gamma_0}} \right) = \frac{1}{2} \left( 1 - \sqrt{1 + \frac{-1}{1+\gamma_0}} \right) \]

Then, we use

\[ \sqrt{1+x} = 1 + x/2 + \ldots \]

which leads to

\[ P_e \approx 1/4\gamma_0 \quad \text{for large } \gamma_0 \]
BER vs. SNR (cont.)

BER \( (= P_e) \)

- Frequency-selective channel (equalization or Rake receiver)
- Frequency-selective channel (no equalization)
- “BER floor”
- Flat fading channel
- AWGN channel (no fading)

\[ P_e \approx \frac{1}{4} \gamma_0 \] means a straight line in log/log scale
BER vs. SNR, summary

<table>
<thead>
<tr>
<th>Modulation</th>
<th>$P_e (\gamma)$</th>
<th>$P_e$</th>
<th>$P_e$ (for large $\gamma_0$)</th>
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<tbody>
<tr>
<td>2-PSK</td>
<td>$Q(\sqrt{2\gamma})$</td>
<td>$\frac{1}{2} \left(1 - \frac{\gamma_0}{\sqrt{1+\gamma_0}}\right)$</td>
<td>$1/4\gamma_0$</td>
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<tr>
<td>DPSK</td>
<td>$e^{-\gamma}/2$</td>
<td>$1/(2\gamma_0 + 2)$</td>
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<td>2-FSK (coh.)</td>
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<td>$e^{-\gamma/2}/2$</td>
<td>$1/(\gamma_0 + 2)$</td>
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Better performance through diversity

Diversity $\Leftrightarrow$ the receiver is provided with multiple copies of the transmitted signal. The multiple signal copies should experience *uncorrelated fading* in the channel.

In this case the probability that *all* signal copies fade simultaneously is reduced dramatically with respect to the probability that a *single* copy experiences a fade.

As a rough rule:

$$P_e \text{ is proportional to } \frac{1}{\gamma_0^L}$$

Diversity of $L$:th order
Selection diversity vs. signal combining

Selection diversity: Signal with best quality is selected.

Equal Gain Combining (EGC)
Signal copies are combined coherently:

\[ Z_{EGC} = \sum_{i=1}^{L} a_i e^{j\phi_i} e^{-j\phi_i} = \sum_{i=1}^{L} a_i \]

Maximum Ratio Combining (MRC, best SNR is achieved)
Signal copies are weighted and combined coherently:

\[ Z_{MRC} = \sum_{i=1}^{L} a_i e^{j\phi_i} a_i e^{-j\phi_i} = \sum_{i=1}^{L} a_i^2 \]
Selection diversity performance

We assume:

(a) uncorrelated fading in diversity branches
(b) fading in \(i\):th branch is Rayleigh distributed
(c) \(\Rightarrow\) SNR is exponentially distributed:

\[
p(\gamma_i) = \frac{1}{\gamma_0} e^{-\gamma_i / \gamma_0}, \quad \gamma_i \geq 0.
\]

PDF

Probability that SNR in branch \(i\) is less than threshold \(y\):

\[
P(\gamma_i < y) = \int_{0}^{y} p(\gamma_i) \, d\gamma_i = 1 - e^{-y / \gamma_0}.
\]

CDF
Selection diversity (cont.)

Probability that SNR in every branch (i.e. all $L$ branches) is less than threshold $y$:

$$P(\gamma_1, \gamma_2, \ldots, \gamma_L < y) = \left[ \int_0^y p(\gamma_i) \, d\gamma_i \right]^L = \left[ 1 - e^{-y/\gamma_0} \right]^L.$$

Note: this is true only if the fading in different branches is independent (and thus uncorrelated) and we can write

$$p(\gamma_1, \gamma_2, \ldots, \gamma_L) = p(\gamma_1) p(\gamma_2) \ldots p(\gamma_L).$$
Selection diversity (cont.)

Differentiating the cdf (cumulative distribution function) with respect to $y$ gives the pdf

$$p(y) = L \left[ 1 - e^{-y/\gamma_0} \right]^{L-1} \cdot \frac{e^{-y/\gamma_0}}{\gamma_0}$$

which can be inserted into the expression for average bit error probability

$$P_e = \int_{0}^{\infty} P_e(y) p(y) dy.$$ 

The mathematics is unfortunately quite tedious ...
Selection diversity (cont.)

... but as a general rule, for large $\gamma_0$ it can be shown that

$$P_e \text{ is proportional to } \frac{1}{\gamma_0^L}$$

regardless of modulation scheme (2-PSK, DPSK, 2-FSK).

The largest diversity gain is obtained when moving from $L = 1$ to $L = 2$. The relative increase in diversity gain becomes smaller and smaller when $L$ is further increased.

This behaviour is typical for all diversity techniques.
BER vs. SNR (diversity effect)

- BER ($= P_e$)
- SNR ($= \gamma_0$)
- $L = 1$ (Flat fading channel, Rayleigh fading)
- $L = 4$
- $L = 3$
- $L = 2$
- AWGN channel (no fading)
MRC performance

Rayleigh fading => SNR in $i$:th diversity branch is

$$\gamma_i = \frac{E_b}{N_0} a_i^2 = \frac{E_b}{N_0} \left( x_i^2 + y_i^2 \right)$$

Rayleigh distributed magnitude

Gaussian distributed quadrature components

In case of $L$ uncorrelated branches with same fading statistics, the MRC output SNR is

$$\gamma = \frac{E_b}{N_0} \left( a_1^2 + a_2^2 \ldots + a_L^2 \right) = \frac{E_b}{N_0} \left( x_1^2 + y_1^2 \ldots + x_L^2 + y_L^2 \right)$$
MRC performance (cont.)

The pdf of $\gamma$ follows the \textit{chi-square distribution} with $2L$ degrees of freedom.

Reduces to exponential pdf when $L = 1$

$$p(\gamma) = \frac{\gamma^{L-1}}{\gamma_0^L \Gamma(L)} e^{-\gamma/\gamma_0} = \frac{\gamma^{L-1}}{\gamma_0^L (L-1)!} e^{-\gamma/\gamma_0}$$

Gamma function       Factorial

For 2-PSK, the average BER is

$$P_e = \left(\frac{1-\mu}{2}\right)^L \sum_{k=0}^{L-1} \binom{L-1+k}{k} \left(\frac{1+\mu}{2}\right)^k$$

$$P_e(\gamma) = Q\left(\sqrt{2\gamma}\right)$$

$$\mu = \frac{\sqrt{\gamma_0}/(1+\gamma_0)}$$
MRC performance (cont.)

For large values of average SNR this expression can be approximated by

$$P_e = \left( \frac{1}{4\gamma_0} \right)^L \binom{2L-1}{L}$$

which again is according to the general rule

$$P_e \text{ is proportional to } \frac{1}{\gamma_0^L}.$$
MRC performance (cont.)

The second term in the BER expression does not increase dramatically with $L$:

\[
\binom{2L-1}{L} = \frac{(2L-1)!}{L! \cdot (L-1)!} = 1 \quad L = 1 \\
= 3 \quad L = 2 \\
= 10 \quad L = 3 \\
= 35 \quad L = 4
\]
BER vs. SNR for MRC, summary

For large $\gamma_0 \quad \implies \quad P_e = \left( \frac{1}{k\gamma_0} \right)^L \left( \begin{array}{c} 2L - 1 \\ L \end{array} \right)$

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Why is MRC optimum performance?

Let us investigate the performance of a signal combining method in general using arbitrary weighting coefficients $g_i$.

Signal magnitude and noise energy/bit at the output of the combining circuit:

$$Z = \sum_{i=1}^{L} g_i \cdot a_i$$

$$N_t = N_0 \sum_{i=1}^{L} g_i^2$$

SNR after combining:

$$\gamma = \frac{Z^2 E_b}{N_t} = \frac{E_b (\sum g_i a_i)^2}{N_0 \sum g_i^2}$$
Why is MRC optimum performance? (cont.)

Applying the Schwarz inequality

\[ (\sum g_i a_i)^2 \leq \sum g_i^2 \sum a_i^2 \]

it can be easily shown that in case of equality we must have \( g_i = a_i \) which in fact is the definition of MRC.

Thus for MRC the following important rule applies (the rule also applies to SIR = Signal-to-Interference Ratio):

\[ \gamma = \sum_{i=1}^{L} \gamma_i \]

Output SNR or SIR = sum of branch SNR or SIR values
Matched filter = "full-scale" MRC

Let us consider a single symbol in a narrowband system (without ISI). If the sampled symbol waveform before matched filtering consists of $L+1$ samples

$$r_k, \quad k = 0, 1, 2, \ldots, L$$

the impulse response of the matched filter also consists of $L+1$ samples

$$h_k = r_{L-k}^*$$  \text{Definition of matched filter}

and the output from the matched filter is

$$Z = \sum_{k=0}^{L} h_k r_{L-k} = \sum_{k=0}^{L} r_{L-k}^* r_{L-k} = \sum_{k=0}^{L} |r_k|^2$$  \text{MRC !}
Matched filter = MRC (cont.)

The discrete-time (sampled) matched filter can be presented as a transversal FIR filter:

\[
Z = \sum_{k=0}^{L} |r_k|^2
\]

\[h_0 = r_L^*\]

\[h_1\]

\[h_{L-1}\]

\[h_L\]

\[Z\]