Sampling Theory

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Sampling Dilemma

![Graph showing a waveform amplitude over time](image)
Sampling Dilemma
The Theory

Sampling Theorem

If a signal \( x(t) \) contains no frequency components for frequencies above \( f = W \) hertz, then it is completely described by instantaneous sample values uniformly spaced in time with period \( T_s \leq 1/2W \).

That is, the sampling frequency \( f_s = 1/T_s \) needs to satisfy

\[
f_s \geq 2W.
\]

The frequency \( 2W \) is referred to as the *Nyquist frequency*. 

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Suppose that $x(t)$ is a continuous-time signal. $x[n]$ is the discrete-time signal that consists of samples of $x(t)$ with a sampling period $T_s$. Therefore,

$$x[n] = x(nT_s), \quad -\infty < n < \infty.$$
Suppose that the Fourier transform of $x(t)$ is $X(f)$. That is,

$$X(f) = \mathcal{F}\{x(t)\}.$$

The continuous-time representation of the sampled signal is

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \delta(t - nT_s)$$

$$= x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

where $\delta(t)$ is the Dirac delta function.
The Fourier transform of $x_s(t)$ is $X_s(f)$, which can be calculated as

$$X_s(f) = \mathcal{F}\{x_s(t)\} = X(f) \otimes \mathcal{F}\left\{ \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \right\}$$

$$= X(f) \otimes \left[ f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s) \right]$$

$$= f_s X(f) \otimes \sum_{n=-\infty}^{\infty} \delta(f - nf_s)$$

$$= f_s \sum_{n=-\infty}^{\infty} X(f) \otimes \delta(f - nf_s)$$

$$= f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$$

where, $f_s = 1/T_s$ is the sampling frequency.
Low pass filtering

$x_s(t)$

$f \leq f_s/2$

W

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In order to reconstruct the original signal $x(t)$, we need to pass the sampled signal $x_s(t)$ through an ideal low-pass filter (rectangular function in frequency) to remove the high-frequency replicas.

A prefect $X(f)$ can be extracted by applying the rectangular function for filtering only when

$$f_s/2 \geq W$$

where $W$ is the largest frequency component in signal $x(t)$. □
Sampling rate $f_s$ is smaller than the Nyquist rate $2W$. 