1 Inverse Systems and Deconvolution
In many practical applications we are given an output signal from a system whose characteristics are unknown and we are asked to determine the input signal.
In many practical applications we are given an output signal from a system whose characteristics are unknown and we are asked to determine the input signal.

Channel distortion and a need for a corrective system: Equalizer, Inverse system
In many practical applications we are given an output signal from a system whose characteristics are unknown and we are asked to determine the input signal.

Channel distortion and a need for a corrective system: Equalizer, Inverse system

An inverse system — The corrective system has a frequency response which is basically the reciprocal of the frequency response of the system that caused the distortion.
In many practical applications we are given an output signal from a system whose characteristics are unknown and we are asked to determine the input signal.

Channel distortion and a need for a corrective system: Equalizer, Inverse system

An inverse system — The corrective system has a frequency response which is basically the reciprocal of the frequency response of the system that caused the distortion.

Deconvolution — The inverse system operation that takes \( y(n) \) and produces \( x(n) \).
In many practical applications we are given an output signal from a system whose characteristics are unknown and we are asked to determine the input signal.

Channel distortion and a need for a corrective system: Equalizer, Inverse system

An inverse system — The corrective system has a frequency response which is basically the reciprocal of the frequency response of the system that caused the distortion.

Deconvolution — The inverse system operation that takes $y(n)$ and produces $x(n)$.

System Identification — In short, to find $h(n)$ or $H(\omega)$. 
A system is said to be *invertible* if there is a one-to-one correspondence between its input and output signals.

An invertible system: $\mathcal{T}$

The inverse system: $\mathcal{T}^{-1}$

$$w(n) = \mathcal{T}^{-1}[y(n)] = \mathcal{T}^{-1}\{\mathcal{T}[x(n)]\} = x(n)$$
LTI system $\mathcal{T}$ has impulse response $h(n)$; the inverse system $\mathcal{T}^{-1}$ has impulse response $h_I(n)$.

\[ w(n) = h_I(n) \otimes h(n) \otimes x(n) = x(n) \]

\[ h(n) \otimes h_I(n) = \delta(n) \]

Therefore,

\[ H(z)H_I(z) = 1 \]

\[ H_I(z) = \frac{1}{H(z)} \]
Invertibility of Linear Time-Invariant Systems

LTI system $\mathcal{T}$ has impulse response $h(n)$; the inverse system $\mathcal{T}^{-1}$ has impulse response $h_I(n)$.

$$H_I(z) = \frac{1}{H(z)}$$

If $H(z)$ has a rational system function

$$H(z) = \frac{B(z)}{A(z)}$$

then

$$H_I(z) = \frac{A(z)}{B(z)}$$

- The zeros of $H(z)$ become the poles of the inverse system, and vice versa.
- If $H(z)$ is an FIR system, then $H_I(z)$ is an all-pole system, and vice versa.
Invertibility of Linear Time-Invariant Systems

\[ h(n) \otimes h_I(n) = \delta(n) \]

We assume that the system and its inverse are causal. Then this equation simplifies to

\[ \sum_{k=0}^{n} h(k)h_I(n-k) = \delta(n) \]

For \( n = 0 \), \( h_I(0) = 1/h(0) \).

For \( n \geq 1 \), \( h_I(n) \) can be obtained recursively

\[ h_I(n) = \sum_{k=1}^{n} \frac{h(n)h_I(n-k)}{h(0)}, \quad n \geq 1 \]
Minimum-Phase, Maximum-Phase, and Mixed-Phase Systems

e.g.,

\[ H_1(z) = 1 + \frac{1}{2}z^{-1} \]
\[ H_2(z) = \frac{1}{2} + z^{-1} \]

\[ |H_1(\omega)| = |H_2(\omega)| = \sqrt{\frac{5}{4} + \cos \omega} \]
\[ \angle H_1(\omega) = -\omega + \tan^{-1}\frac{\sin \omega}{0.5 + \cos \omega} \]
\[ \angle H_2(\omega) = -\omega + \tan^{-1}\frac{\sin \omega}{2 + \cos \omega} \]
Minimum-Phase, Maximum-Phase, and Mixed-Phase Systems

\[ \angle H_1(\omega) = -\omega + \tan^{-1} \frac{\sin \omega}{0.5 + \cos \omega} \]

\[ \angle H_2(\omega) = -\omega + \tan^{-1} \frac{\sin \omega}{2 + \cos \omega} \]

Minimum-phase: \[ \angle H(\pi) - \angle H(0) = 0 \]; Maximum-phase:
\[ \angle H(\pi) - \angle H(0) = \pi. \]
Minimum-Phase, Maximum-Phase, and Mixed-Phase Systems

For an FIR system that has $M$ zeros,

$$H(\omega) = b_0(1 - z_1 e^{-j\omega})(1 - z_2 e^{-j\omega}) \cdots (1 - z_M e^{-j\omega})$$

- When all zeros are inside the unit circle, Minimum-phase:
  $$\angle H(\pi) - \angle H(0) = 0;$$
- When all zeros are outside the unit circle, Maximum-phase:
  $$\angle H(\pi) - \angle H(0) = M\pi.$$

If the FIR system with $M$ zeros has some of its zeros inside the unit circle and the remaining zeros outside the unit circle, it is called a mixed-phase system or a nonminimum-phase system.
Minimum-Phase, Maximum-Phase, and Mixed-Phase Systems

Since the derivative of the phase characteristic of the system is a measure of the time delay that signal frequency components undergo in passing through the system,

- a minimum-phase characteristic implies a minimum delay function;
- a maximum-phase characteristic implies that the delay characteristic is also maximum.

Because

\[ |H(\omega)|^2 = H(z)H(z^{-1})|_{z = e^{j\omega}}, \]

if we replace a zero \( z_k \) of the system by its inverse \( 1/z_k \), the magnitude characteristic of the system does not change.

Place zeros inside unit circle for minimum phase.
Minimum-Phase, Maximum-Phase, and Mixed-Phase Systems

Extend to IIR systems that have rational system functions

\[ H(z) = \frac{B(z)}{A(z)} \]

It is minimum-phase, if all its poles and zeros are inside the unit circle.

For a stable and causal system, the system is maximum phase if all the zeros are outside the unit circle.

- A stable pole-zero system that is minimum phase has a stable inverse which is also minimum phase. Why?
- Maximum-phase systems and mixed-phase systems result in unstable inverse systems.
Decomposition of Nonminimum-phase Pole-zero Systems.

Any nonminimum-phase pole-zero system can be expressed as

\[ H(z) = H_{\text{min}}(z)H_{\text{ap}}(z) \]

\( H(z) \) is causal and stable.
\( B(z) = B_1(z)B_2(z) \), where \( B_1(z) \) has all its roots inside the unit circle, \( B_2(z) \) has all its roots outside the unit circle.

Then,

\[ H_{\text{min}}(z) = \frac{B_1(z)B_2(z^{-1})}{A(z)} \]
\[ H_{\text{ap}}(z) = \frac{B_2(z)}{B_2(z^{-1})} \]

\( H_{\text{ap}}(z) \) is a stable, all-pass, maximum-phase system.
Group delay: \( \tau_g(\omega) = \tau_{g, \text{min}}(\omega) + \tau_{g, \text{ap}}(\omega) \)
System Identification and Deconvolution

\[ y(n) = h(n) \otimes x(n) \]

\[ H(z) = \frac{Y(z)}{X(z)} \]

The system can be identified uniquely if it is known causal. Alternatively, if the system is causal,

\[ y(n) = \sum_{k=0}^{n} h(k)x(n - k), \quad n \geq 0 \]

hence, recursively, we have

\[ h(0) = \frac{y(0)}{x(0)} \]

\[ h(n) = \frac{y(n) - \sum_{k=0}^{n-1} h(k)x(n - k)}{x(0)}, \quad n \geq 1 \]
The crosscorrelation method is an effective and practical method for system identification.

\[ r_{yx}(m) = \sum_{k=0}^{\infty} h(k) r_{xx}(m-k) = h(m) \otimes r_{xx}(m) \]

\[ S_{yx}(\omega) = H(\omega) S_{xx}(\omega) = H(\omega)|X(\omega)|^2 \]

Therefore,

\[ H(\omega) = \frac{S_{yx}(\omega)}{S_{xx}(\omega)} = \frac{S_{yx}(\omega)}{|X(\omega)|^2} \]