ELC 4351: Digital Signal Processing

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Discrete-time Signals and Systems

1. Discrete-time Signals
2. Discrete-time Systems
3. Analysis of Discrete-time Linear Time-Invariant Systems
4. Implementation of Discrete-time Systems
5. Correlation of Discrete-time Signals
Elementary Discrete-time Signals

1. Unit sample sequence

\[ \delta(n) = \begin{cases} 
1, & n = 0 \\
0, & n \neq 0 
\end{cases} \]

2. Unit step signal

\[ u(n) = \begin{cases} 
1, & n \geq 0 \\
0, & n < 0 
\end{cases} \]

3. Unit ramp signal

\[ u_r(n) = \begin{cases} 
n, & n \geq 0 \\
0, & n < 0 
\end{cases} \]

4. Exponential signal

\[ x(n) = a^n = (re^{j\theta})^n = r^n e^{j\theta n} \]
Classification of Discrete-time Signals

Energy signals vs. power signals

Energy: \( E = \sum_{n=-\infty}^{\infty} |x(n)|^2. \)

If \( E \) is finite, \( 0 < E < \infty \), \( x(n) \) is energy signal.

Power: \( P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 = \lim_{N \to \infty} \frac{1}{2N+1} E_N. \)

\( E \) finite \( \Rightarrow \) \( P = 0. \)

If \( P \) is finite, \( 0 < P < \infty \), \( x(n) \) is power signal.
Periodic signals vs. aperiodic signals

\( x(n) \) is periodic with period \( N > 0 \) iff

\[
x(n + N) = x(n), \quad \forall n.
\]

The smallest \( N \) is the fundamental period.

E.g., \( x(n) = A \sin(2\pi fn), \quad f = \frac{k}{N}. \)

Power: \( P = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2. \)

Therefore, periodic signals are power signals.
Symmetric (even) vs. antisymmetric (odd) signals

Even: \( x(-n) = x(n) \)
Odd: \( x(-n) = -x(n) \)

Any signal can be expressed as a sum of an even signal and an odd signal.

\[
x(n) = x_e(n) + x_o(n)
\]

Proof.
\[
x_e(n) = \frac{1}{2}[x(n) + x(-n)] \quad \text{and} \quad x_o(n) = \frac{1}{2}[x(n) - x(-n)].
\]
Simple Manipulations of Discrete-time Signals

Time-delay: $TD_k[x(n)] = x(n - k), \ k > 0$.

Folding: $FD[x(n)] = x(-n)$.

Amplitude scaling: $y(n) = Ax(n), \ -\infty < n < \infty$.

Sum: $y(n) = x_1(n) + x_2(n)$.

Product: $y(n) = x_1(n)x_2(n).$ (sample-to-sample basis)
Discrete-time System

\[ y(n) = \mathcal{T}[x(n)] \]
Input-Output Description of Systems

\[ x(n) \rightarrow^T y(n) \quad y(n) = T[x(n)] \]

For example, an accumulator:

\[
y(n) = \sum_{k=-\infty}^{n} x(k)
\]

\[
= x(n) + x(n-1) + x(n-2) + \cdots + x(n-n) + x(n)
\]

\[
= \sum_{k=-\infty}^{n-1} x(k) + x(n)
\]

\[
= y(n-1) + x(n)
\]

Initially relaxed at \( n_0 \): \( y(n_0 - 1) = 0 \).
Block Diagram Representation of Discrete-time Systems

Adder

\[ y(n) = x_1(n) + x_2(n) \]

Constant Multiplier

\[ y(n) = a x(n) \]

Signal Multiplier

\[ y(n) = x_1(n) x_2(n) \]
Unit Delay Element

\[ x(n) \xrightarrow{z^{-1}} y(n) = x(n-1) \]

Unit Advance Element

\[ x(n) \xrightarrow{z} y(n) = x(n+1) \]
Classification of Discrete-time Systems

Static vs. dynamic systems

**Static (memoryless):**

\[
  y(n) = \alpha x(n) \\
  y(n) = n^2 x(n) + \beta x^2(n)
\]

**Dynamic:**

\[
  y(n) = x(n) + 3x(n - 1) \\
  y(n) = \sum_{k=0}^{\infty} x(n - k)
\]
Time-invariant vs. time-variant systems

Time-invariant:

\[ x(n) \rightarrow^T y(n) \quad \text{implies} \quad x(n - k) \rightarrow^T y(n - k). \]

\[ y(n, k) = T[x(n - k)] = y(n - k) \]
Classification of Discrete-time Systems

**Linear vs. nonlinear systems**

Linear system iff

\[
\mathcal{T}[\alpha_1 x_1(n) + \alpha_2 x_2(n)] = \alpha_1 \mathcal{T}[x_1(n)] + \alpha_2 \mathcal{T}[x_2(n)]
\]

Superposition: Scaling (multiplicative) property + Additive property

**Figure 2.2.9** Graphical representation of the superposition principle. \( \mathcal{T} \) is linear if and only if \( y(n) = y'(n) \).
Classification of Discrete-time Systems

Causal vs. noncausal systems

Causal system iff

\[ y(n) = \mathcal{T}[x(n), x(n - 1), x(n - 2), \cdots] \]
Stable vs. unstable systems

Bounded input - bounded output (BIBO) stable iff

\[ |x(n)| \leq M_x < \infty \Rightarrow |y(n)| \leq M_y < \infty, \forall n. \]
Interconnection of Discrete-time Systems

Cascade:

\[ y(n) = T_2[T_1[x(n)]] , \quad T_c = T_2T_1 \]

In general, \( T_2T_1 \neq T_1T_2 \).

Parallel:

\[ y(n) = T_1[x(n)] + T_2[x(n)] , \quad T_p = T_1 + T_2 \]
For LTI systems, a general form of the input-output relationship.

\[ y(n) = - \sum_{k=1}^{N} a_k y(n - k) + \sum_{k=0}^{M} b_k x(n - k) \]

A difference equation
Techniques for Analysis of Linear Time-invariant Systems

We use \( x(n) = \sum_k c_k x_k(n) \), where \( x_k(n) \) are the elementary signal components.

Suppose that \( y_k(n) = T[x_k(n)] \), we have

\[
    y(n) = T[x(n)] = T \left[ \sum_k c_k x_k(n) \right] \\
    = \sum_k c_k T[x_k(n)] = \sum_k c_k y_k(n)
\]

It is chosen that, e.g.,

\[
    x_k = e^{j\omega_k n}, \quad k = 0, 1, \ldots, N - 1.
\]

where, \( \omega_k = \frac{2\pi k}{N} \). \( \{\omega_k\} \) are harmonically related. \( \frac{2\pi}{N} \) is the fundamental frequency.
Resolution of a Discrete-time Signal into Impulses

We choose

\[ x_k(n) = \delta(n - k) \]

\[ x(n)\delta(n - k) = x(k)\delta(n - k) \]

Therefore,

\[ x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n - k) \]

\[ = \sum_{k=-\infty}^{\infty} x(k)x_k(n) \]
Resolution of a Discrete-time Signal into Impulses

(a) $x(n)$

(b) $\delta(n-k)$

(c) $x(k) \delta(n-k)$
Response of LTI Systems to Arbitrary Inputs

\[ h(n, k) \equiv \mathcal{T}[\delta(n - k)] \]

We use \( x(n) = \sum_{k=-\infty}^{\infty} x(k) \delta(n - k) \).

\[
y(n) = \mathcal{T}[x(n)] = \sum_{k=-\infty}^{\infty} x(k) \mathcal{T}[\delta(n - k)] = \sum_{k=-\infty}^{\infty} x(k) h(n, k)
\]

Time-invariant: \( h(n) = \mathcal{T}[\delta(n)] \Rightarrow h(n, k) = h(n - k) = \mathcal{T}[\delta(n - k)] \)

\[
y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n - k)
\]
The convolution sum

\[ y(n) = x(n) \otimes h(n) \]

\[ = \sum_{k=-\infty}^{\infty} x(k) h(n - k) \]

\[ = \sum_{k=-\infty}^{\infty} h(k) x(n - k) \]

\[ = h(n) \otimes x(n) \]
Identity and Shifting Properties

\[ y(n) = x(n) \otimes \delta(n) = x(n) \]

\[ y(n - k) = x(n) \otimes \delta(n - k) = x(n - k) \]
Properties of Convolution and Interconnection of Systems

**Commutative Law**

\[ x(n) \otimes h(n) = h(n) \otimes x(n) \]

**Associative Law**

\[ [x(n) \otimes h_1(n)] \otimes h_2(n) = x(n) \otimes [h_1(n) \otimes h_2(n)] \]
Distributive Law

\[ x(n) \otimes [h_1(n) + h_2(n)] = x(n) \otimes h_1(n) + x(n) \otimes h_2(n) \]
Causal Linear Time-Invariant Systems

\[ y(n_0) = \sum_{k=-\infty}^{\infty} h(k)x(n_0 - k) \]

\[ = \sum_{k=0}^{\infty} h(k)x(n_0 - k) + \sum_{k=-\infty}^{-1} h(k)x(n_0 - k) \]

\[ \tilde{y}(n) \]

The second part \( \tilde{y}(n) \) depends on the future (w.r.t. \( n_0 \)) inputs \( x(n_0 + 1), x(n_0 + 2), \ldots \) It has to be zero for a causal LTI system.

Therefore, the impulse response of the system must satisfy the condition

\[ h(n) = 0, \quad n < 0 \]

An LTI system is causal iff its impulse response is zero for negative values of \( n \).
\[ h(n) = 0, \quad n < 0 \]

\[
y(n) = \sum_{k=0}^{\infty} h(k)x(n - k)
\]

\[
y(n) = \sum_{k=-\infty}^{n} x(k)h(n - k)
\]
Stability of Linear Time-Invariant Systems

If \( x(n) \) is bounded, \( |x(n)| \leq M_x < \infty, \forall n. \)
If \( y(n) \) is bounded, \( |y(n)| \leq M_y < \infty, \forall n. \)

\[
y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)
\]

\[
|y(n)| = \left| \sum_{k=-\infty}^{\infty} h(k)x(n-k) \right| \leq M_x \sum_{k=-\infty}^{\infty} |h(k)| \leq M_x \sum_{k=-\infty}^{\infty} |h(k)|
\]
We observe that, for $|y(n)| < \infty$, a sufficient condition is

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

It turns out this condition is not only sufficient but also necessary to ensure the stability of the system.

A LTI system is stable iff its impulse response is absolutely summable.
A finite-duration impulse response (FIR) system has an impulse response that is zero outside of some finite time interval.

\[
h(n) = 0, \quad n < 0 \quad \text{and} \quad n \geq M
\]

An infinite-duration impulse response (IIR) system has an infinite-duration impulse response.

\[
y(n) = \sum_{k=0}^{M-1} h(k)x(n - k)
\]

where causality is assumed.
Implementation of Discrete-time Systems

For example, a first-order system described by the linear constant-coefficient difference equation.

\[ y(n) = -a_1 y(n - 1) + b_0 x(n) + b_1 x(n - 1) \]

(1) Use a nonrecursive system followed by a recursive system:

\[ v(n) = b_0 x(n) + b_1 x(n - 1) \]
\[ y(n) = -a_1 y(n - 1) + v(n) \]

(2) Use a recursive system followed by a nonrecursive system:

\[ w(n) = -a_1 w(n - 1) + x(n) \]
\[ y(n) = b_0 w(n) + b_1 w(n - 1) \]
Implementation of Discrete-time Systems

(a)

(b)
Implementation of Discrete-time Systems

\[ y(n) = -\sum_{k=1}^{N} a_k y(n - k) + \sum_{k=0}^{M} b_k x(n - k) \]

(1) Direct form I structure:

\[ v(n) = \sum_{k=0}^{M} b_k x(n - k) \]

\[ y(n) = -\sum_{k=1}^{N} a_k y(n - k) + v(n) \]
Direct Form I Structure

\[ x(n) \xrightarrow{b_0} \xrightarrow{\mathbf{z}^{-1}} v(n) \xrightarrow{\mathbf{z}^{-1}} y(n) \]

- \( b_0 \) and \( v(n) \) are input and output, respectively.
- \( \mathbf{z}^{-1} \) represents the delay operator.
- \( b_1, b_2, \ldots, b_{M-1}, b_M \) are the direct coefficients.
- \( a_1, a_2, \ldots, a_{N-1}, a_N \) are the feedback coefficients.
Implementation of Discrete-time Systems

\[ y(n) = - \sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \]

(2) Direct form II structure:

\[ w(n) = - \sum_{k=1}^{N} a_k w(n-k) + x(n) \]

\[ y(n) = \sum_{k=0}^{M} b_k w(n-k) \]
Direct Form II Structure

\[ x(n) \xrightarrow{+} \circ \xrightarrow{-a_1} z^{-1} \xrightarrow{+} \circ \xrightarrow{-a_2} z^{-1} \xrightarrow{+} \circ \xrightarrow{-a_3} z^{-1} \ldots \xrightarrow{-a_{N-2}} z^{-1} \xrightarrow{+} \circ \xrightarrow{-a_{N-1}} z^{-1} \xrightarrow{+} \circ \xrightarrow{-a_N} z^{-1} \xrightarrow{+} b_0 \xrightarrow{+} y(n) \]

\( z^{-1} \) indicates a unit delay in the system.
Crosscorrelation of sequences $x(n)$ and $y(n)$ is a sequence $r_{xy}(l)$ defined as

$$
r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l), \quad l = 0, \pm 1, \pm 2, \ldots$$

$$
= \sum_{n=-\infty}^{\infty} x(n+l)y(n), \quad l = 0, \pm 1, \pm 2, \ldots
$$

where index $l$ is the time shift or lag.

$$r_{xy}(l) = r_{yx}(-l)$$

$$r_{xy}(l) = x(l) \otimes y(-l)$$
Correlation of Discrete-time Signals

Autocorrelation of sequence $x(n)$ is a sequence $r_{xx}(l)$ defined as

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)x(n-l), \; l = 0, \pm 1, \pm 2, \ldots$$

$$= \sum_{n=-\infty}^{\infty} x(n+l)x(n), \; l = 0, \pm 1, \pm 2, \ldots$$

where index $l$ is the time shift or lag.

$$r_{xx}(l) = r_{xx}(-l)$$

$$r_{xx}(l) = x(l) \otimes x(-l)$$
Properties of Autocorrelation and Crosscorrelation Sequences

\[ |r_{xx}(l)| \leq r_{xx}(0) = E_x \]
\[ |r_{xy}(l)| \leq \sqrt{r_{xx}(0)r_{yy}(0)} = \sqrt{E_xE_y} \]

Normalized autocorrelation sequence:

\[ \rho_{xx}(l) = \frac{r_{xx}(l)}{r_{xx}(0)}, \quad |\rho_{xx}(l)| \leq 1 \]

Normalized crosscorrelation sequence:

\[ \rho_{xy}(l) = \frac{r_{xy}(l)}{\sqrt{r_{xx}(0)r_{yy}(0)}}, \quad |\rho_{xy}(l)| \leq 1 \]
Crosscorrelation:

\[ r_{xy}(l) = \frac{1}{N} \sum_{n=0}^{N-1} x(n)y(n - l) \]

Autocorrelation:

\[ r_{xx}(l) = \frac{1}{N} \sum_{n=0}^{N-1} x(n)x(n - l) \]
Example: Correlation is used to identify periodicity in an observed physical signal that is corrupted by random noise/interference.

\[ y(n) = x(n) + w(n) \]

We observe \( M \) samples of \( y(n) \), where \( M \gg N \).

\[
\begin{align*}
 r_{yy}(l) &= \frac{1}{M} \sum_{n=0}^{M-1} y(n)y(n - l) \\
 &= \frac{1}{M} \sum_{n=0}^{M-1} [x(n) + w(n)][x(n - l) + w(n - l)] \\
 &= r_{xx}(l) + r_{xw}(l) + r_{wx}(l) + r_{ww}(l)
\end{align*}
\]
Example: Identify a hidden periodicity in the Wölfer sunspot numbers in the 100-year period 1770-1869.
Crosscorrelation between the output and the input signal is

\[ r_{yx}(l) = y(l) \otimes x(-l) = h(l) \otimes [x(l) \otimes x(-l)] \]
\[ = h(l) \otimes r_{xx}(l) \]

Autocorrelation of the output signal is

\[ r_{yy}(l) = y(l) \otimes y(-l) \]
\[ = [h(l) \otimes x(l)] \otimes [h(-l) \otimes x(-l)] \]
\[ = [h(l) \otimes h(-l)] \otimes [x(l) \otimes x(-l)] \]
\[ = r_{hh}(l) \otimes r_{xx}(l) \]

The autocorrelation \( r_{hh}(l) \) of the impulse response \( h(n) \) exists if the system is stable.