ELC 4351: Digital Signal Processing

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The z-Transform and Its Application to the Analysis of LTI Systems

1. The z-Transform
   - The Direct z-Transform
   - The Inverse z-Transform

2. Properties of the z-Transform
Laplace-Transform: Continuous-time signals and LTI systems

z-Transform: Discrete-time signals and LTI systems
The direct z-transform is a power series.

**Transform Equation**

\[ X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \]

where, \( z \) is a complex variable.

It can be expressed as \( X(z) = \mathcal{Z}\{x(n)\} \) or \( x(n) \leftarrow z \ X(z) \).

The region of convergence (ROC) of \( X(z) \) is the set of all values of \( z \) for which \( X(z) \) attains a finite value.
Discussion on ROC

\[ z = re^{j\theta}. \quad r = |z| \text{ and } \theta = \angle z. \]

Transformation Equation

\[
X(z) = \sum_{n=-\infty}^{\infty} x(n)r^{-n}e^{-j\theta n}
\]

In the ROC, \(|X(z)| < \infty\).

Therefore

\[
|X(z)| = \left| \sum_{n=-\infty}^{\infty} x(n)r^{-n}e^{-j\theta n} \right|
\leq \sum_{n=-\infty}^{\infty} |x(n)r^{-n}e^{-j\theta n}|
\]

\[
= \sum_{n=-\infty}^{\infty} |x(n)r^{-n}|
\]
\[ |X(z)| \leq \sum_{n=-\infty}^{\infty} |x(n)r^{-n}| < \infty \]

\( |X(z)| \) is finite if the sequence \( x(n)r^{-n} \) is absolutely summable.
Discussion on ROC

\[ |X(z)| \leq \sum_{n=-\infty}^{\infty} |x(n)r^{-n}| \]

\[ = \sum_{n=-\infty}^{-1} |x(n)r^{-n}| + \sum_{n=0}^{\infty} |x(n)r^{-n}| \]

\[ = \sum_{n=1}^{\infty} |x(-n)r^{n}| + \sum_{n=0}^{\infty} \left| \frac{x(n)}{r^{n}} \right| \]

finite: \( r \) small enough  
finite: \( r \) large enough

In general, ROC: \( r_2 < r < r_1 \)
Discussion on ROC

ROC: $r_2 < r < r_1$
### Discussion on ROC

<table>
<thead>
<tr>
<th>Signal</th>
<th>ROC</th>
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</thead>
<tbody>
<tr>
<td><strong>Finite-Duration Signals</strong></td>
<td></td>
</tr>
<tr>
<td>Causal</td>
<td>Entire z-plane except z = 0</td>
</tr>
<tr>
<td>Anticausal</td>
<td>Entire z-plane except z = ∞</td>
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<tr>
<td>Two-sided</td>
<td>Entire z-plane except z = 0 and z = ∞</td>
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<tr>
<td><strong>Infinite-Duration Signals</strong></td>
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<tr>
<td>Causal</td>
<td></td>
</tr>
<tr>
<td>Anticausal</td>
<td></td>
</tr>
<tr>
<td>Two-sided</td>
<td>r_2 &lt;</td>
</tr>
</tbody>
</table>
Unilateral z-Transform

Transformation Equation

\[ X^+(z) = \sum_{n=0}^{\infty} x(n)z^{-n} \]
The Inverse z-Transform

**Transformation Equation**

\[ x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \]

where \( C \) denotes the closed contour in the ROC of \( X(z) \), taken in a counterclockwise direction.
Properties of the z-Transform

**Linearity**

If $x_1(n) \leftrightarrow z X_1(z)$ and $x_2(n) \leftrightarrow z X_2(z)$, then

$$x(n) = \alpha_1 x_1(n) + \alpha_2 x_2(n) \leftrightarrow z X(z) = \alpha_1 X_1(z) + \alpha_2 X_2(z)$$

for any constants $\alpha_1$ and $\alpha_2$.

**Time shifting**

If $x(n) \leftrightarrow z X(z)$, then

$$x(n - k) \leftrightarrow z z^{-k} X(z)$$

The ROC of $z^{-k} X(z)$ is the same as that of $X(z)$ except for $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$. 
Properties of the z-Transform

Scaling in the z-domain

If \( x(n) \overset{z}{\leftrightarrow} X(z) \), ROC: \( r_1 < |z| < r_2 \), then

\[ a^n x(n) \overset{z}{\leftrightarrow} X(a^{-1}z), \quad \text{ROC:} \quad |a| r_1 < |z| < |a| r_2 \]

for any constants \( a \), real or complex.

Time reversal

If \( x(n) \overset{z}{\leftrightarrow} X(z) \), ROC: \( r_1 < |z| < r_2 \), then

\[ x(-n) \overset{z}{\leftrightarrow} X(z^{-1}), \quad \text{ROC:} \quad \frac{1}{r_2} < |z| < \frac{1}{r_1} \]
Differentiation in the z-domain

If $x(n) \leftrightarrow^z X(z)$, then

$$nx(n) \leftrightarrow^z -z \frac{dX(z)}{dz}$$
Properties of the z-Transform

Convolution of two sequences

If \( x_1(n) \xrightarrow{z} X_1(z) \) and \( x_2(n) \xrightarrow{z} X_2(z) \), then

\[
x(n) = x_1(n) \otimes x_2(n) \xrightarrow{z} X(z) = X_1(z)X_2(z)
\]

The ROC of \( X(z) \) is at least the intersection of that for \( X_1(z) \) and \( X_2(z) \).

Correlation of two sequences

If \( x_1(n) \xrightarrow{z} X_1(z) \) and \( x_2(n) \xrightarrow{z} X_2(z) \), then

\[
r_{x_1x_2}(l) = \sum_{n=-\infty}^{\infty} x_1(n)x_2(n - l) \xrightarrow{z} R_{x_1x_2}(z) = X_1(z)X_2(z^{-1})
\]

The ROC of \( R(z) \) is at least the intersection of that for \( X_1(z) \) and \( X_2(z^{-1}) \).
Properties of the z-Transform

Multiplication of two sequences

If \( x_1(n) \overset{z}{\leftrightarrow} X_1(z) \) and \( x_2(n) \overset{z}{\leftrightarrow} X_2(z) \), then

\[
x(n) = x_1(n)x_2(n) \overset{z}{\leftrightarrow} X(z) = \frac{1}{2\pi j} \oint_C X_1(\nu)X_2\left(\frac{z}{\nu}\right)\nu^{-1}d\nu
\]

where \( C \) is a closed contour that encloses the origin and lies within the ROC common to both \( X_1(\nu) \) and \( X_2(1/\nu) \).
Parseval’s relation

If $x_1(n)$ and $x_2(n)$ are complex-valued sequences, then

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j} \oint_C X_1(\nu)X_2^*(\frac{1}{\nu^*})\nu^{-1} d\nu$$
The Initial Value Theorem

If \( x(n) \) is causal, i.e. \( x(n) = 0 \) for \( n < 0 \), then

\[
x(0) = \lim_{z \to \infty} X(z)
\]

Proof.

\[
X(z) = \sum_{n=0}^{\infty} x(n)z^{-n}
\]

\[
= x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots
\]

As \( z \to \infty \), \( z^{-n} \to 0 \) when \( n = 1, 2, \ldots \), therefore \( X(z) \to x(0) \).