1 Linear Time-Invariant Systems as Frequency-Selective Filters
A LTI system performs a type of discrimination or filtering among the various frequency components at its input.

The nature of this filtering action is determined by the frequency response characteristics $H(\omega)$. 
By proper selection of the coefficients $a_k$’s and $b_k$’s, we can design frequency-selective filters.

These filters pass signals with frequency components in some bands while they attenuate signals containing frequency components in other frequency bands.
Ideal Filter Characteristics

A filter with frequency response

\[
H(\omega) = \begin{cases} 
Ce^{-j\omega n_0}, & \omega_1 < \omega < \omega_2 \\
0, & \text{otherwise}
\end{cases}
\]

where \( C \) and \( n_0 \) are constants.

\[
Y(\omega) = X(\omega)H(\omega) = CX(\omega)e^{-j\omega n_0}
\]

\[
y(n) = Cx(n - n_0)
\]

- The filter output is simply a delayed and amplitude-scaled version of the input signal.
- A pure delay is usually tolerable and is not considered a distortion of the signal. Neither is amplitude scaling.

Therefore, ideal filters have a linear phase characteristic within their passband, that is,
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A filter with frequency response

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where $C$ and $n_0$ are constants.

Ideal filters have a linear phase characteristic within their passband, that is,

$$\Theta(\omega) = -\omega n_0$$

*Group delay* of the filter

$$\tau_g(\omega) = -\frac{d\Theta(\omega)}{d\omega}$$

Linear phase = group delay is constant. In this case, all frequency components of the input signal undergo the same time delay.
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- Design some simple digital filters by the placement of poles and zeros in the z-plane.
- The location of poles and zeros affects the frequency response characteristics of the system.
The basic principle underlying the pole-zero placement method:

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  However, zeros can be placed anywhere in the z-plane.
- All complex zeros and poles must occur in complex-conjugate pairs in order for the filter coefficients to be real.
The system function:

\[ H(z) = b_0 \frac{\prod_{k=1}^{M} (1 - z_k z^{-1})}{\prod_{k=1}^{N} (1 - p_k z^{-1})} \]

Usually, \( b_0 \) is selected such that \( |H(\omega_0)| = 1 \). \( \omega_0 \) in the passband of the filter.
\( N \geq M \).
Lowpass, Highpass, and Bandpass Filters

- Design of lowpass digital filters: the poles should be placed near the unit circle at points corresponding to low frequencies (near $\omega = 0$) and zeros should be placed near or on the unit circle at points corresponding to high frequencies (near $\omega = \pi$).
- Design of highpass digital filters: The opposite.
A Simple Lowpass-to-Highpass Filter Transformation

Frequency translation of $\pi$ rad:

$$H_{hp}(\omega) = H_{lp}(\omega - \pi)$$

Therefore,

$$h_{hp}(n) = e^{j\pi n} h_{lp}(n) = (-1)^n h_{lp}(n)$$

e.g., Lowpass filter by difference eqn.

$$y(n) = - \sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

A highpass filter can be derived: (How?)

$$y(n) = - \sum_{k=1}^{N} (-1)^k a_k y(n-k) + \sum_{k=0}^{M} (-1)^k b_k x(n-k)$$
A digital resonator is a special two-pole bandpass filter with the pair of complex-conjugate poles located near the unit circle.

The filter has a large magnitude response (i.e., it resonates) in the vicinity of the pole location.

The angular position of the pole determines the resonant frequency of the filter.

Digital resonators are useful in many applications, including bandpass filtering and speech generation.
A resonant peak at or near $\omega = \omega_0$,

$$p_{1,2} = re^{\pm j\omega_0}, \quad 0 < r < 1$$

We can select up to two zeros –

One choice is to locate the zeros at the origin.

The other choice is to locate a zero at $z = 1$ and a zero at $z = -1$. This choice completely eliminates the response of the filter at frequencies $\omega = 0$ and $\omega = \pi$. 
Digital Resonator

Digital resonator with zeros at the origin:

\[ H(\omega) = \frac{b_0}{(1 - re^{j\omega_0}e^{-j\omega})(1 - re^{-j\omega_0}e^{-j\omega})} \]

We select \( b_0 \) so that \( |H(\omega_0)| = 1 \).

\[ H(\omega_0) = \frac{b_0}{(1 - re^{j\omega_0}e^{-j\omega_0})(1 - re^{-j\omega_0}e^{-j\omega_0})} = \frac{b_0}{(1 - r)(1 - re^{-j2\omega_0})} \]

\[ |H(\omega_0)| = \frac{b_0}{(1 - r)\sqrt{1 + r^2 - 2r \cos 2\omega_0}} = 1 \]

\[ b_0 = (1 - r)\sqrt{1 + r^2 - 2r \cos 2\omega_0} \]
Digital Resonator

Digital resonator with zeros at the origin:

\[
H(\omega) = \frac{b_0}{(1 - re^{j\omega_0}e^{-j\omega})(1 - re^{-j\omega_0}e^{-j\omega})}
\]

\[
|H(\omega_0)| = \frac{b_0}{U_1(\omega)U_2(\omega)}
\]

\[
\angle H(\omega) = 2\omega - \Phi_1(\omega) - \Phi_2(\omega)
\]

\[
U_1(\omega) = \sqrt{1 + r^2 - 2r \cos(\omega_0 - \omega)}
\]

\[
U_2(\omega) = \sqrt{1 + r^2 - 2r \cos(\omega_0 + \omega)}
\]
Digital Resonator

\[ U_1(\omega) = \sqrt{1 + r^2 - 2r \cos(\omega_0 - \omega)} \]

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\[ \min_{\omega} U_1(\omega)U_2(\omega) \implies \omega_r = \cos^{-1}\left( \frac{1 + r^2}{2r} \cos \omega_0 \right) \]
Digital resonator with zeros $z = 1$ and $z = -1$:

$$H(\omega) = b_0 \frac{(1 - e^{-j\omega})(1 + e^{-j\omega})}{(1 - re^{j\omega_0} e^{-j\omega})(1 - re^{-j\omega_0} e^{-j\omega})}$$

$$|H(\omega)| = b_0 \frac{\sqrt{2(1 - \cos 2\omega)}}{U_1(\omega) U_2(\omega)}$$

The actual resonant frequency is altered.
All-Pass Filters

\[ |H(\omega)| = 1, \quad 0 \leq \omega \leq \pi \]

e.g.,

1. a pure delay system \( H(z) = z^{-k} \).

2. \[
H(z) = \frac{\sum_{k=0}^{N} a_k z^{-N+k}}{\sum_{k=0}^{N} a_k z^{-k}}, \quad a_0 = 1
\]

where \( A(z) = \sum_{k=0}^{N} a_k z^{-k} \).

\[
|H(\omega)|^2 = H(z)H(1/z) \bigg|_{z=e^{j\omega}} = 1
\]
If $z_0$ is a pole of $H(z)$, then $1/z_0$ is a zero of $H(z)$.

The poles and zeros are reciprocals of one another.
All-Pass Filters

All-pass filter with real coefficients:

\[
H_{ap}(z) = \prod_{k=1}^{N_R} \frac{z^{-1} - \alpha_k}{1 - \alpha_k z^{-1}} \prod_{k=1}^{N_C} \frac{(z^{-1} - \beta_k)(z^{-1} - \beta_k^*)}{(1 - \beta_k z^{-1})(1 - \beta_k^* z^{-1})}
\]

where there are \( N_R \) real poles and zeros and \( N_C \) complex-conjugate pairs of poles and zeros.

For causal and stable systems, \( 1 < \alpha_k < 1 \) and \( |\beta| < 1 \).

Q: What is all-pass filter for?
A: All-pass filters find application as phase equalizers. When placed in cascade with a system that has an undesired phase response, a phase equalizer is designed to compensate for the poor phase characteristics of the system and therefore to produce an overall linear-phase response.