1. Frequency Analysis of Continuous-Time Signals
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2. Frequency Analysis of Discrete-Time Signals
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A linear combination of harmonics (harmonically related complex exponentials):

**Synthesis Equation**

\[ x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_0 t} \]

**Analysis Equation**

\[ c_k = \frac{1}{T_p} \int_{T_p} x(t) e^{-j2\pi kF_0 t} dt \]

where, the fundamental period is \( T_p = 1/F_0 \).
A linear combination of cosine functions, if signal $x(t)$ is real:

**Synthesis Equation**

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos 2\pi k F_0 t - b_k \sin 2\pi k F_0 t)$$

where

$$a_0 = c_0$$
$$a_k = 2|c_k| \cos \theta_k$$
$$b_k = 2|c_k| \sin \theta_k$$
$$c_k = |c_k| e^{i \theta_k}$$
The Dirichlet conditions guarantee that $x(t)$ and its Fourier series representation are equal at any value of $t$:

1. $x(t)$ has a finite number of discontinuities in any period.
2. $x(t)$ contains a finite number of maxima and minima during any period.
3. $x(t)$ is absolutely integrable in any period, i.e. $\int_{T_p} |x(t)| dt < \infty$. 
A periodic signal has a finite average power

\[
P_x = \frac{1}{T_p} \int_{T_p} |x(t)|^2 dt
\]

\[
= \frac{1}{T_p} \int_{T_p} x(t) x^*(t) dt
\]

\[
= \frac{1}{T_p} \int_{T_p} x(t) \sum_{k=-\infty}^{\infty} c_k^* e^{-j2\pi kF_0 t} dt
\]

\[
= \sum_{k=-\infty}^{\infty} c_k^* \left[ \frac{1}{T_p} \int_{T_p} x(t) e^{-j2\pi kF_0 t} dt \right]
\]

\[
= \sum_{k=-\infty}^{\infty} |c_k|^2 \quad \text{(Parseval’s Relation)}
\]

\[
P_x = a_0^2 + \frac{1}{2} \sum_{k=1}^{\infty} (a_k^2 + b_k^2)
\]
The Fourier Transform for Continuous-Time Aperiodic Signals

Going from periodic signal to aperiodic signal, we make the period $T_p \to \infty$.

$$x(t) = \lim_{T_p \to \infty} x_p(t)$$

$$x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_0 t}, \quad F_0 = \frac{1}{T_p}$$

$$c_k = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x(t) e^{-j2\pi kF_0 t} dt$$

$$= \frac{1}{T_p} \int_{-\infty}^{\infty} x(t) e^{-j2\pi kF_0 t} dt$$

$$= X(F)$$
The Fourier Transform for Continuous-Time Aperiodic Signals

We write $F \triangleq kF_0 = k/T_p$ and $\Delta F \triangleq F_0 = 1/T_p$.

As $T_p \to \infty$, $\Delta F = dF$. Therefore

$$x_p(t) = \frac{1}{T_p} \sum_{k=-\infty}^{\infty} X(F) e^{j2\pi kF_0 t}$$

$$= \sum_{k=-\infty}^{\infty} X(k\Delta F) e^{j2\pi kF_0 t} \Delta F$$

$$x(t) = \lim_{T_p \to \infty} x_p(t)$$

$$= \lim_{\Delta F \to 0} \sum_{k=-\infty}^{\infty} X(k\Delta F) e^{j2\pi kF_0 t} \Delta F$$

$$= \int_{-\infty}^{\infty} X(F) e^{j2\pi F t} dF$$
The Fourier Transform for Continuous-Time Aperiodic Signals

Synthesis Equation (Inverse Transform)

\[ x(t) = \int_{-\infty}^{\infty} X(F)e^{j2\pi Ft} dF \]

Analysis Equation (Direct Transform)

\[ X(F) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi Ft} dt \]
Energy Density Spectrum of Aperiodic Signals

Signal Energy: \( E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt \)

\[
E_x = \int_{-\infty}^{\infty} x(t)x^*(t)dt \\
= \int_{-\infty}^{\infty} x(t)dt \left[ \int_{-\infty}^{\infty} X^*(F)e^{-j2\pi F t} dF \right] \\
= \int_{-\infty}^{\infty} X^*(F)dF \left[ \int_{-\infty}^{\infty} x(t)e^{-j2\pi F t} dt \right] \\
= \int_{-\infty}^{\infty} X^*(F)X(F)dF \\
= \int_{-\infty}^{\infty} |X(F)|^2 dF
\]
Energy Density Spectrum of Aperiodic Signals

Parseval’s Relation

\[ E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(F)|^2 dF \]
Energy Density Spectrum:

\[ S_{xx}(F) \overset{\Delta}{=} |X(F)|^2 \]

Therefore, \( S_{xx}(F) \geq 0 \), for all \( F \).

If signal \( x(t) \) is real, \( |X(-F)| = |X(F)| \) and \( \angle X(-F) = -\angle X(F) \). It follows that

\[ S_{xx}(-F) = S_{xx}(F) \]
The Fourier Series of Discrete-Time Periodic Signals

\( x(n) \) is periodic with period \( N \). That is, \( x(n) = x(n + N) \) for all \( n \).

A linear combination of \( N \) harmonically related exponents:

**Synthesis Equation**

\[
x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}
\]

**Analysis Equation**

\[
c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}
\]
The Fourier series coefficients \( \{ c_k \} \) is a periodic sequence with fundamental period \( N \) (when extended outside the range \([0, N - 1]\)).

\[
c_{k+N} = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi(k+N)n/N}
\]

\[
= \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}
\]

\[
= c_k
\]

The spectrum of \( x(n) \) is a periodic sequence with period \( N \).
A linear combination of cousin functions, if signal $x(n)$ is real:

**Synthesis Equation**

$$x(n) = a_0 + 2 \sum_{k=1}^{L} (a_k \cos(2\pi kn/N) - b_k \sin(2\pi kn/N))$$

where

- $a_0 = c_0$
- $a_k = 2|c_k| \cos \theta_k$
- $b_k = 2|c_k| \sin \theta_k$
- $L = \begin{cases} N/2 & \text{if } N \text{ is even} \\ (N - 1)/2 & \text{if } N \text{ is odd} \end{cases}$
The average power of a discrete-time periodic signal with period $N$:

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x(n)x^*(n)$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x(n) \left( \sum_{k=0}^{N-1} c_k^* e^{-j2\pi kn/N} \right)$$

$$= \sum_{k=0}^{N-1} c_k^* \left[ \frac{1}{N} \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \right]$$

$$= \sum_{k=0}^{N-1} |c_k|^2$$
Energy over a signal period:

\[ E_N = \sum_{n=0}^{N-1} |x(n)|^2 = N \sum_{k=0}^{N-1} |c_k|^2 \]

If \( x(n) \) is real, \( c_k^* = c_{-k} \). Equivalently, \( |c_{-k}| = |c_k| \) and \( -\angle c_{-k} = \angle c_k \).
The Fourier Transform of Discrete-Time Aperiodic Signals

**Analysis Equation**

\[ X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}, \quad \omega \in [-\pi, \pi) \text{ or } \omega \in [0, 2\pi) \]

**Synthesis Equation**

\[ x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega \]

\(X(\omega)\) is periodic with period \(2\pi\):

\[ X(\omega + 2\pi k) = \sum_{n=-\infty}^{\infty} x(n) e^{-j(\omega+2\pi k)n} \]

\[ = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = X(\omega) \]
Convergence of the Fourier Transform

\[ X_N(\omega) = \sum_{n=-N}^{N} x(n)e^{-j\omega n} \]

Uniform convergence:

\[
\lim_{N \to \infty} \{\sup_{\omega} |X(\omega) - X_N(\omega)|\} = 0, \quad \text{for all } \omega
\]

Uniform convergence is guaranteed if \( \sum_{n=-\infty}^{\infty} |x(n)| < \infty \).

Mean-square convergence:

\[
\lim_{N \to \infty} \int_{-\pi}^{\pi} |X(\omega) - X_N(\omega)|^2 d\omega = 0, \quad \text{for all } \omega
\]

Mean-square convergence is for finite-energy signals \( \sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty \).
Energy Density Spectrum of Aperiodic Signals

The energy of a discrete-time signal $x(n)$:

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} x(n)x^*(n)$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega)e^{-j\omega n} d\omega \right]$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega) \left[ \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$
Energy Density Spectrum:

\[ S_{xx}(\omega) \triangleq |X(\omega)|^2 \]

If \( x(n) \) is real, \( X^*(\omega) = X(-\omega) \). Equivalently, \( |X(-\omega)| = |X(\omega)| \) and \( \angle X(-\omega) = -\angle X(\omega) \). It follows that

\[ S_{xx}(-\omega) = S_{xx}(\omega) \]
z-Transform

\[ X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}; \quad \text{ROC: } r_2 < |z| < r_1 \]

z in polar form: \( z = re^{j\omega} \). We have

\[ X(z) = \sum_{n=-\infty}^{\infty} [x(n)r^{-n}]e^{-j\omega n} \]

If \( X(z) \) converges for \( |z| = 1 \),

\[ X(z) \big|_{z=e^{j\omega}} = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \]
Relationship of the Fourier Transform to the z-Transform

\[ X(z) \big|_{z=e^{j\omega}} = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \]

Fourier transform can be viewed as the z-transform of the sequence evaluated on the unit circle.

If \( X(z) \) does not converge in the region \(|z| = 1\), the Fourier transform \( X(\omega) \) does not exist.
Frequency-Domain Classification of Signals: The Concept of Bandwidth

Power (energy) density spectrum concentration

\{ \begin{align*}
& \text{low-frequency} \\
& \text{high-frequency} \\
& \text{bandpass}
\end{align*} \}

Bandwidth — a quantitative measure

Suppose a continuous-time signal has 90% of its power (energy) density spectrum in range $F_1 < F < F_2$. The 90% bandwidth of the signal is $F_2 - F_1$. 
Frequency-Domain Classification of Signals: The Concept of Bandwidth

Narrowband: \( F_2 - F_1 \ll \frac{F_1 + F_2}{2} \) (median frequency)
Wideband: Otherwise

Bandlimited: \( X(F) = 0 \) for \( |F| > B \)
\( X(\omega) = 0 \) for \( \omega_0 < |\omega| < \pi \)

No signal can be time-limited and band-limited simultaneously. (Reciprocal relationship)